99 A

Excellent!

MATH 112, Exam I

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Instructor: John Ramsay

Name Rebecca Ross

There are 7 problems. On all problems, *show all work* (justifying your answers is part of the exam) and follow instructions carefully. A list of potentially useful formulas can be found on the last page. If anything is unclear, please ask!

1. In each of the following, find $\frac{dy}{dx}$:

(5 pts each)

a) $y = \ln(\tan x)$

$$y' = \frac{1}{\tan x} \cdot \sec^2 x$$ ✓

b) $y = \tan^{-1}\left(\frac{2-x}{3}\right)$

$\tan^{-1}\left(\frac{2}{3} - \frac{x}{3}\right)$

$$y' = \frac{1}{1 + \left(\frac{2-x}{3}\right)^2} \cdot \left(-\frac{1}{3}\right)$$ ✓

c) $e^{(x^2+y)} = \cos(x)$

$$\ln e^{(x^2+y)} = \ln \cos(x)$$

$$x^2 + y = \ln \cos(x)$$

$$y = \ln \cos(x) - x^2$$

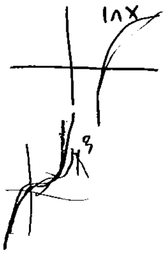
(-1)

$$y' = \frac{1}{\cos x}(-\sin x) - 2x$$

2. Compute the following limits (Justify your answers.):

(5 pts each)

a) $\lim_{x\to\infty} \frac{\ln x}{x^3}$ $\frac{\infty}{\infty}$ (LR) $\frac{\frac{1}{x}}{3x^2}$ $\frac{1}{x} \cdot \frac{1}{3x^2} = \frac{1}{3x^3} = \frac{1}{\infty} = \boxed{0}$ ✓



b) $\lim_{x\to 0^+} (2x)^x$ 0^0 $\ln (2x)^x = \overset{0 \cdot -\infty}{x \ln(2x)} = \frac{\ln(2x)}{\frac{1}{x}} = \frac{-\infty}{\infty}$ (LR) $\frac{\frac{1}{\cancel{2}x} \cdot \cancel{2}}{-\frac{1}{\cancel{2}x^2}}$

$$= \frac{1}{\cancel{x}} \cdot \frac{-2x^{\cancel{2}}}{1} = -2x = -2(0) = 0$$

$$e^0 = \boxed{1}$$ ✓

x^{-1}

$\frac{x^{-2}}{-2} = -\frac{1}{2x^2}$

24

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1$$

$e \ln e$

3. (12 pts)

a) Define the constant *e* as it was defined in class and in the lab.

$a^x \ln a$

$$\lim_{h\to 0} \frac{x^h - x}{h} = 1$$

$e^x \ln e$

$e^x (1)$

$$\frac{1}{h} \lim_{x\to 0} x^h - x$$

-3

b) Prove either that $\frac{d}{dx}\left[a^x\right] = a^x \ln a$ or that $\frac{d}{dx}\left[\tan^{-1}(x)\right] = \frac{1}{x^2+1}$.

$$\frac{d}{dx}[a^x] = a^x \ln a$$

$a^x = y$

$x = \log_a y$

$1 = \frac{1}{y \ln a} \frac{dy}{dx}$

$y \ln a = \frac{dy}{dx}$

$a^x \ln a = \frac{dy}{dx}$ ✓

✓

4. Prove both of the following:
(12 pts)

a) $\frac{d}{dx}[\ln x] = \frac{1}{x}$

$\ln x = y$

$e^y = x$

$e^y \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{e^y}$

$\frac{dy}{dx} = \frac{1}{x}$ ✓

✓

b) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$

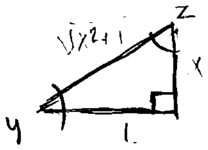
$\tan^{-1} x = y$ $\quad \cot^{-1} x = z$

$\tan y = x$ $\quad \cot z = x$

$z + y = \frac{\pi}{2}$

✓

∂



$z + y + 90 = 180$

$z + y = 90 \longrightarrow 90° = \frac{\pi}{2}$

$\therefore\ z + y = \frac{\pi}{2}$ ✓

5. Evaluate the following integrals. You must show the steps of your work. (7 pts each)

a) $\int_0^1 xe^x dx$

$u = x$ $\quad dv = e^x\, dx$

$du = dx$ $\quad v = e^x dx$

$uv - \int v\, du$

$xe^x - \int e^x dx$

$xe^x - e^x \Big]_0^1$

$1e \cancel{-} e - (0 - 1)$

$\boxed{1}$

✓

b) $\int \frac{\ln(2x)}{x} dx$

$u = \ln(2x)$

$du = \frac{1}{2x} \cdot 2 \;\; dx$

$du = \frac{1}{x} dx$

$x\,du = dx$

$\int \frac{u}{\cancel{x}} \cancel{x}\, du$

$= \frac{u^2}{2}$

$= \boxed{\frac{(\ln 2x)^2}{2} + C}$

✓

c) $\int_0^{1/2} \frac{x^2}{\sqrt{1-x^2}} dx$

$x = \sin\theta$

$dx = \cos\theta \;\, d\theta$

~~$x = \sin\theta$~~

~~$\frac{1}{2} = \sin\theta$~~

~~$\arcsin \frac{1}{2} = \theta$~~

~~$\frac{\pi}{6} = \theta$~~

~~$x = \sin\theta$~~

~~$0 = \sin\theta$~~

~~$\arcsin 0 = \theta$~~

~~$0 = \theta$~~

$\int_0^{1/2} \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}} \cdot \cancel{\cos\theta}\, d\theta$

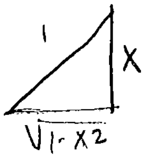
$\sqrt{\cos^2\theta}$

$\cancel{\cos\theta}$

$\int_0^{1/2} \sin^2\theta \;\; d\theta$

$s^2 + x^2 = 1$

$\sqrt{1-x^2}$



$\arcsin(x) = \theta$

$\int_0^{1/2} \sin^2\theta \;\; d\theta$ ✓

$\int \frac{1 - \cos 2\theta}{2}$ $\qquad + C$

$\int \frac{1}{2} - \frac{1}{2}\int \cos 2\theta$

$u = 2\theta$

$\frac{du}{2} = d\theta$

$\int \frac{\theta}{2} - \frac{1}{4}\sin 2\theta$ ✓

$\frac{\arcsin x}{2}$

$-\frac{1}{\cancel{4}\,2}\,\cancel{2}\sin\theta\cos\theta$

$\frac{\arcsin x}{2} - \frac{1}{2}(x)(\sqrt{1-x^2})\Big]_0^{1/2}$

$\frac{\pi}{12} - \frac{1}{2}(\frac{1}{2})(\sqrt{.75}) -$

$\cancel{0 + 0}$

$\frac{\pi}{12} - \frac{1}{4}(\sqrt{.75})$ ✓ o.k.

$= \boxed{\text{[illegible]}}$ ✓

$x^2 - \cancel{x + x} - 1$

$(x+1)(x-1)$

d) $\int \frac{x^2}{x^2-1}\,dx$ $= \int 1 + \int \frac{1}{x^2-1}$

$$x^2-1 \overline{)\, \cancel{x^2}} \quad 1 + \frac{1}{x^2-1}$$

$-\cancel{x^2} + 1$

-1

x $\quad \int \frac{1}{(x+1)(x-1)}$ $\quad \frac{A}{x+1} + \frac{B}{x-1}$

$A(x-1) + B(x+1)$

$\underline{Ax} - A + \underline{Bx} + B$

$A+B=0$ $\quad -A+B=1$

$B = 1 + A$

$A + (1+A) = 0$

$2A + 1 = 0$

$A = -\frac{1}{2}$ $\quad \frac{1}{2} + B = 1$

$B = \frac{1}{2}$

$$x - \frac{1}{2}\int \frac{1}{x+1} + \frac{1}{2}\int \frac{1}{x-1}$$

$$\boxed{x - \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C}$$

✓

e) $\int \frac{1}{x^{3/2} + x^{1/2}}\,dx$

~~x = tanθ~~

~~dx = sec²θ dθ~~

$u = \sqrt{x+1}$

$\int \frac{1}{\sqrt{x}\,(x+1)}$

$u^2 = x+1$ $\quad u^2 - 1 = x$

$\int \frac{1}{\sqrt{u^2-1}\; u^2} \cdot 2u\,du$ $\quad 2u\,du = dx$

$$\int \frac{2}{u\sqrt{u^2-1}}\,du \quad u = \sec\theta$$

$du = \sec\theta\tan\theta\, d\theta$

$\theta = \text{arcsec}\, u$

$\theta = \text{arcsec}\sqrt{x+1}$

$$\int \frac{2}{\sec\theta\sqrt{\sec^2\theta - 1}} \cdot \sec\theta\tan\theta\, d\theta$$

$\tan\theta$

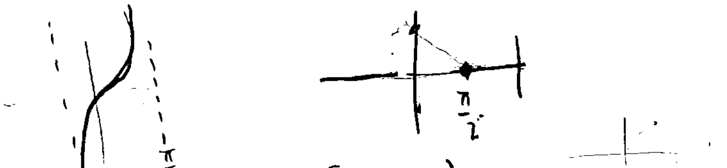
$$= 2\int \frac{\sec\theta\tan\theta}{\sec\theta\tan\theta}\, d\theta = 2\int 1\, d\theta$$

$= 2\theta$

$= \boxed{2\,\text{arcsec}\sqrt{x+1} + C}$

✓

14



6. Consider the function $f(x) = (\tan x)^{(\cos x)}$

Compute $f'(x)$ and $\lim_{x \to \pi/2^-} f(x)$

(11 pts)

$f(x) = (\tan x)^{\cos x}$

$y = (\tan x)^{(\cos x)}$

$\ln y = \ln(\tan x)^{\cos x}$

$\ln y = \cos x \ln(\tan x)$

$\frac{1}{y} y' = -\sin x \cdot \ln(\tan x) + \cos x \cdot \frac{1}{\tan x} \cdot \sec^2 x$

$$\boxed{y' = \left(-\sin(x)\ln(\tan x) + \cos(x)\frac{\sec^2 x}{\tan x}\right)\left(\tan x^{(\cos x)}\right)}$$ ✓

$\lim_{x \to \frac{\pi}{2}^-} (\tan x)^{(\cos x)}$

∞^0 $\nearrow \infty$

$= \ln \tan x^{\cos x}$

$= \cos x \ln \tan x \quad = \frac{\ln(\tan x)}{\frac{1}{\cos x}} \sec x \quad \frac{\infty}{\infty}$ Ⓛ

$0 \cdot \infty$

$\frac{1}{0} = \infty$

$\frac{\frac{1}{\tan x}\cdot \sec^2 x}{\sec x \tan x} \cdot \frac{\sec^2 x}{\tan x} \cdot \frac{1}{\cancel{\sec} x \tan x}$

$= \frac{\sec x}{\tan^2 x} = \frac{\frac{1}{\cos}}{\frac{\sin^2 x}{\cos^2 x}} = \frac{1}{\cancel{\cos}} \cdot \frac{\cos^{\cancel{2}} x}{\sin^2 x}$

$\frac{\cos x}{\sin^2 x} = \frac{0}{1} = 0$ ✓

$e^0 = \boxed{1}$

7. (5 pts + 3 bonus pts)
   a. Which century is known as the "heroic" century because of the incredible advancement of science during that time?

      17th century ✓

   b. Name three important mathematicians of this century and briefly identify their contribution to the calculus.

      ① napier + brigs - logs
      ② descartes - analytic geometry ✓
      ③ newton - fundamental theory of calculus

19